

The group of inertial automorphisms of an abelian group

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dedicated to Martin L. Newell

Abstract

We study the group $\text{IAut}(A)$ generated by the inertial automorphisms of an abelian group A , that is, automorphisms γ with the property that each subgroup H of A has finite index in the subgroup generated by H and $H\gamma$. Clearly, $\text{IAut}(A)$ contains the group $\text{FAut}(A)$ of finitary automorphisms of A , which is known to be locally finite. In a previous paper, we showed that $\text{IAut}(A)$ is (locally finite)-by-abelian. In this paper, we show that $\text{IAut}(A)$ is also metabelian-by-(locally finite). In particular, $\text{IAut}(A)$ has a normal subgroup Γ such that $\text{IAut}(A)/\Gamma$ is locally finite and Γ' is an abelian periodic subgroup whose all subgroups are normal in Γ . In the case when A is periodic, $\text{IAut}(A)$ results to be abelian-by-(locally finite) indeed, while in the general case it is not even (locally nilpotent)-by-(locally finite). Moreover, we provide further details about the structure of $\text{IAut}(A)$ in some other cases for A .

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1 Introduction

An endomorphism γ of an abelian group $(A, +)$ is called *inertial endomorphism* if and only if $(H + H\gamma)/H$ is finite for each subgroup H (see [10], [12]). An inertial endomorphism which is bijective is called *inertial automorphism*. This definition can be seen as a generalization of the notion of *finitary automorphism* of A , that is, an automorphism γ acting as the identity map on a subgroup of finite index in A (see [2], [18]). Since A is

abelian, this condition is clearly equivalent to $A(\gamma - 1)$ being finite. Note that we regard abelian groups as right modules over their endomorphism ring and reserve the letter A for abelian groups, which are *additively written*.

The concept of inertial endomorphism of an abelian group A may be used as a tool in the study of *inert* subgroups of possibly non-abelian groups (see [9], [11]). Recall that a subgroup is called inert if it is commensurable with its conjugates (see [1], [16]), where subgroups H and K are called *commensurable* if and only if $H \cap K$ has finite index in both H and K .

In this paper we study the group $\text{IAut}(A)$ generated by all inertial automorphisms of an abelian group A . Recall that in [7] (resp. [10]) we gave a description of inertial automorphisms (resp. endomorphisms) of an abelian group, while the ring of inertial endomorphisms of A was featured in [8]. In particular, from [10], we have:

- $\text{IAut}(A)$ consists of products $\gamma_1 \gamma_2^{-1}$ where γ_1 and γ_2 are both inertial automorphisms,
- $\text{IAut}(A)$ is locally (central-by-finite),
- $\text{IAut}(A)$ is abelian modulo its subgroup $\text{FAut}(A)$ of finitary automorphisms.

Recall that $\text{FAut}(A)$ is known to be locally finite ([18]).

Note that the above definitions of $\text{IAut}(G)$ and $\text{FAut}(G)$ make sense even if the underlying group G is not abelian, and that $\text{FAut}(G) \leq \text{IAut}(G)$ in any case. Also, in [2] it has been shown that the group $\text{FAut}(G)$ of finitary automorphisms of any group G is both abelian-by-(locally finite) and (locally finite)-by-abelian.

Question. *Is $\text{IAut}(A)$, the group generated by all inertial automorphisms of an abelian group A , abelian-by-(locally finite)?*

Our main results, in Sect. 3, can be summarized as follows:

- Theorem A gives a complete description of the group $\text{IAut}(A)$, when A is periodic;
- Corollary A asserts that the answer to our question is in the positive, when A is periodic;
- Theorem B asserts that, for any abelian group A , the group $\text{IAut}(A)$ has a metabelian subgroup Γ such that $\text{IAut}(A)/\Gamma$ is locally finite and each subgroup of Γ' is normal in Γ ;
- Corollary B asserts that the answer is in the negative for $A = \mathbb{Z}(p^\infty) \oplus \mathbb{Q}_p$;
- Theorem C describes the group $\text{IAut}(A)$ in some cases when A is non-periodic.

In our investigations, we shall look for abelian normal subgroups Σ of $\text{IAut}(A)$ such that the automorphisms induced by $\text{IAut}(A)$ via conjugation on Σ are inertial, see Theorem C. Thus, in Sect. 4, we highlight the role played by stability groups with respect to finitary and inertial automorphisms of A . In Sect. 5 we treat the case when A is periodic (Theorem A). In Sect. 6 we treat the remaining cases (Theorems B and C).

For undefined terminology, notation and basic facts we refer to [14] or [15]. In particular, $\pi(n)$ denotes the set of prime divisors of $n \in \mathbb{Z}$. If π is a set of primes, then A_π , $T(A)$ and $D(A)$ denote the unique maximum π -subgroup, the torsion subgroup and the divisible subgroup of the abelian group A , respectively. By exponent $m = \exp(A)$ of a p -group A we mean the smallest m such that $p^m A = 0$, or $m = \infty$ if A is unbounded. Furthermore, \mathbb{Q}_p is the additive group of rational numbers whose denominator is a power of the prime p and $\mathbb{Z}(p^\infty) := \mathbb{Q}_p/\mathbb{Z}$. Also, $r_0(A)$ denotes the torsion-free rank of A , i.e. the cardinality of a maximal \mathbb{Z} -independent subset of A .

If $A_i \leq A$ and $\gamma \in \text{Aut}(A)$, as usual we denote by $\gamma|_{A_i}$ the restriction of γ to A_i . If we have $A = A_1 \oplus A_2$ and $\gamma|_{A_i} = \gamma_i \in \text{Aut}(A_i)$ (for $i = 1, 2$), we write $\gamma = \gamma_1 \oplus \gamma_2$.

Commutators are calculated in the holomorph group $A \rtimes \text{Aut}(A)$. Moreover, if φ is an endomorphism of the additive abelian group A and $a \in A$, we use the notation $[a, \varphi] := a\varphi - a = a(\varphi - 1)$.

2 Preliminaries

It is convenient to recall that an automorphism leaving every subgroup invariant is usually called a *power automorphism*. Then the group $\text{PAut}(A)$ of power automorphisms of an abelian group A can be described as follows (see [15]).

If A is a p -group and $\alpha = \sum_{i=0}^{\infty} \alpha_i p^i$ (with $0 \leq \alpha_i < p$) is an invertible p -adic, we define, with an abuse of notation, the power automorphism α (that we will also call *multiplication* by α) by setting $a\alpha := (\sum_{i=0}^{k-1} \alpha_i p^i)a$, for any $a \in A$ of order p^k . In this way, we have defined an action on A of the group \mathcal{U}_p of units of the ring of p -adic integers, whose image is $\text{PAut}(A)$. If A has infinite exponent, then this action is faithful and $\text{PAut}(A)$ is isomorphic to \mathcal{U}_p . Otherwise, if $e := \exp(A) < \infty$, then the kernel of this action is $\{\alpha \in \mathcal{U}_p \mid \alpha \equiv 1 \pmod{p^e}\}$ and $\text{PAut}(A)$ is isomorphic to the group of units of $\mathbb{Z}(p^e)$.

If A is any periodic abelian group, then $\text{PAut}(A)$ is the cartesian product of all the $\text{PAut}(A_p)$ where A_p is the p -component of A . If A is non-periodic, then $\text{PAut}(A) = \{\pm 1\}$.

According to [10], an automorphism γ is called an (invertible) *multiplication* of A if and only if it is a power automorphism of A , if A is periodic, or -when A is non-periodic- there exist coprime integers m, n such that $(na)\gamma = ma$, for each $a \in A$. In the latter case, we have $mnA = A$ and $A_{\pi(mn)} = 0$ and -with an abuse of notation- we will write $\gamma = m/n$. We warn that we are using the word “multiplication” in a way different from [14]. Invertible multiplications of A form a subgroup which is a central subgroup of $\text{Aut}(A)$.

If $r_0(A) < \infty$, from [10] we have that $\text{IAut}(A)$ contains the group of all invertible multiplications. In this case, $\text{IAut}(A)$ consists of inertial automorphisms only. Furthermore, $\text{IAut}(A)$ is the kernel of the setwise action of $\text{Aut}(A)$ on the quotient of the lattice of the subgroups of A with respect to commensurability (which is a lattice congruence indeed, since A is abelian).

If $r_0(A) = \infty$, then the above kernel is the subgroup of $\text{IAut}(A)$ consisting of so-called *almost-power* automorphisms of A , that is, automorphisms γ such that every subgroup contains a γ -invariant subgroup of finite index. This group was introduced in [13] to study generalized soluble groups in which subnormal subgroups are normal-by-finite (or core-finite, according to the terminology of [3] and [5]).

We recall now some other facts that will be used in the sequel. They follow from Theorem 3 of [7], Proposition 2.2 and Theorem A of [10].

Lemma 2.1. *Let γ be an automorphism of an abelian group A .*

- 1) *If $r_0(A) = \infty$, then γ is inertial if and only if there are a subgroup A_0 of finite index in A and an integer m such that $\gamma = m$ on A_0 .*
- 2) *If $0 < r_0(A) < \infty$, then γ is inertial if and only if there are a torsion-free γ -finitely generated γ -subgroup V such that A/V is periodic, a rational number m/n (with m, n coprime integers) such that $\gamma = m/n$ on V and A_π is bounded, where $\pi := \pi(mn)$. In particular A/A_π is π -divisible.*
- 3) *If A is periodic, then γ inertial if and only if γ is inertial on each p -component of A and acts as a power automorphism on all but finitely many of them.*
- 4) *If A is a p -group, then γ is inertial if and only if either γ acts as an invertible multiplication (that is as a power automorphism) on a subgroup A_0 of finite index in A or (critical case) $0 \neq D := D(A)$ has finite rank, A/D is infinite bounded and there is a subgroup A_1 of finite index in A such that γ acts as invertible multiplication (by possibly different p -adics) on both A_1/D and D . □*

For further instances of inertial automorphisms see Lemma 6.1.

3 Main results

Our first result is about periodic abelian groups.

Theorem A. *Let A be a periodic abelian group. Then there is a subgroup Δ of $\text{IAut}(A)$ which is direct product of finite abelian groups and such that*

$$\text{IAut}(A) = \text{PAut}(A) \cdot \text{FAut}(A) \cdot \Delta$$

where Δ is trivial, if A is reduced.

Moreover, there are a set π of primes and subgroups Σ, Ψ of $\text{IAut}(A)$ such that Σ is an abelian π' -group with bounded primary components and

$$\text{FAut}(A) \cdot \Delta = \text{FAut}(A_\pi) \times (\Sigma \rtimes \Psi)$$

where the automorphisms induced by Ψ via conjugation on Σ are inertial and this action is faithful.

Corollary A. *If A is a periodic abelian group, then $\text{IAut}(A)$ is central-by-(locally finite).*

With an abuse of notation, in Theorem A we regard $\text{FAut}(A_\pi)$ as naturally embedded in $\text{FAut}(A)$. For details in the case A is a p -group see Proposition 5.1 below.

In the next theorem we answer our question in the non-periodic case. We reduce to study the subgroup $\text{IAut}_1(A)$ consisting of *inertial automorphisms of A that act as the identity map on $A/T(A)$* . Actually, applying results from [7], we have that the above introduced group of *almost-power automorphisms* of A is

$$\text{IAut}_1(A) \times \{\pm 1\}.$$

In the case when $r_0(A) = \infty$, we have $\text{IAut}_1(A) = \text{FAut}(A)$ by Lemma 2.1.(1).

We will also consider a group $Q(A)$ of particular inertial automorphisms of A , which is contained in the center of $\text{Aut}(A)$ and is naturally isomorphic to the multiplicative group of rational numbers generated by -1 and primes p such that A/A_p is p -divisible and A_p is either bounded or finite according as $r_0(A)$ is finite or not (see Lemma 6.1 below for details).

Theorem B. *Let A be a non-periodic abelian group. Then there is a subgroup $Q(A)$, which is isomorphic to a multiplicative group of rational numbers, such that*

$$\text{IAut}(A) = \text{IAut}_1(A) \times Q(A)$$

Moreover there is a normal subgroup Γ of $\text{IAut}_1(A)$ such that:

- i) $\text{IAut}_1(A)/\Gamma$ is locally finite;*
- ii) the derived subgroup Γ' of Γ is a periodic abelian group and each subgroup of Γ' is normal in Γ*

Corollary B. *If A is an abelian group, then $\text{IAut}(A)$ is metabelian-by-(locally finite). However, $\text{IAut}(\mathbb{Z}(p^\infty) \oplus \mathbb{Z})$ is not nilpotent-by-(locally finite).*

Note that if A is torsion-free, then $\text{IAut}(A) = Q(A)$ is abelian, as in Theorem 2 of [7]. Further, in the statement of Theorem B one may take Γ to be the subgroup of $\text{IAut}_1(A)$ consisting of inertial automorphisms acting by multiplication on $T(A)$. Unfortunately this subgroup need not be nilpotent, as in Corollary B. On the other hand, groups with property (ii) in Theorem B above have been studied under the name of KI-groups in a series of papers (see [17]).

The next theorem considers cases in which A splits on its torsion subgroup. For details see Propositions 6.3 and 6.4.

Theorem C. *If A is an abelian group with $r_0(A) < \infty$ and either $T := T(A)$ is bounded or A/T is finitely generated, then there are subgroup Σ and Γ_1 of $\text{IAut}_1(A)$ such that*

$$\text{IAut}_1(A) = \Sigma \rtimes \Gamma_1$$

where Σ is a periodic abelian group, $\Gamma_1 \simeq \text{IAut}(T)$ and the automorphisms induced by Γ_1 via conjugation on Σ are inertial

When A/T is not finitely generated, it may happen that A has very few inertial automorphisms, since from Proposition 6.2 and Lemma 2.1.(2) we have

$$\text{IAut}(\mathbb{Z}(p^\infty) \oplus \mathbb{Q}_{(p)}) = \{\pm 1\}.$$

However, in the general case, the group $\text{IAut}_1(A)$ may be large, see Remark 6.5.

4 Finitary automorphisms and Stability groups

We state now some basic facts that perhaps are already known (see also [4]). If $X \leq A$, we denote by $\text{St}(A, X)$ the stability group of the series $A \geq X \geq 0$, that is, the set of $\gamma \in \text{Aut}(A)$ such that $X \geq [A, \gamma] := A(\gamma - 1)$ and $[X, \gamma] = 0$. When X is a characteristic subgroup of A , each $\gamma \in \text{Aut}(A)$ acts via conjugation on the abelian normal subgroup $\Sigma := \text{St}(A, X)$ of $\text{Aut}(A)$, according to the rule $\sigma \mapsto \gamma^{-1}\sigma\gamma =: \sigma^\gamma$ for each $\sigma \in \Sigma$. Similarly, γ acts on the additive group $\text{Hom}(A/X, X)$ of homomorphisms $A/X \rightarrow X$ by a corresponding formula, i.e. $\varphi \mapsto \gamma_{|A/X}^{-1}\varphi\gamma$ where $\varphi \in \text{Hom}(A/X, X)$ and $\gamma_{|A/X}$ denotes the group isomorphism induced by γ on A/X . With an abuse of notation, we denote by $\sigma - 1$ the well-defined homomorphism $\bar{a} \in A/X \mapsto a\sigma - a \in X$.

Fact 4.1. *The map $\mathcal{H} : \sigma \in \text{St}(A, X) \mapsto (\sigma - 1) \in \text{Hom}(A/X, X)$ is an isomorphism of (right) $\text{Aut}(A)$ -modules, that is, for each $\gamma \in \text{Aut}(A)$ we have*

$$\sigma^\gamma = \gamma^{-1}(\sigma - 1)\gamma + 1 \quad \square$$

By this argument we have two technical lemmas. For the first one see [6] .

Lemma 4.2. *Let A be an abelian group, $\sigma, \gamma \in \text{Aut}(A)$ and $m_1, m_2 \in \mathbb{Z}$. If σ stabilizes a series $0 \leq A_1 \leq A$, where $\gamma = m_1$ on A_1 and $\gamma^{-1} = m_2$ on A/A_1 , then $\sigma^\gamma = \sigma^{m_1 m_2}$. \square*

Our next lemma deals with the case when A splits on X and will be used several times. In such a condition, once fixed a direct decomposition $A = X \oplus K$, we have an embedding $\text{Aut}(K) \rightarrow \text{Aut}(A)$ given by $\gamma \mapsto 1 \oplus \gamma$. Note that, if $\Gamma \triangleleft \text{Aut}(A)$, then one can consider $\text{St}_\Gamma(A, X) := \text{St}(A, X) \cap \Gamma$ which is $\text{Aut}(A)$ -isomorphic to a submodule of $\text{Hom}(A/X, X)$. The proof of the lemma is straightforward.

Lemma 4.3. *Let $A = X \oplus K$, where X is a Γ -subgroup, $\Gamma \leq \text{Aut}(A)$, $\zeta : A/X \leftrightarrow K$ the natural isomorphism, $\Sigma := \text{St}_\Gamma(A, X)$, $\Gamma_1 := \{\gamma|_X \oplus 1 \mid \gamma \in \Gamma\}$ and $\Gamma_2 := \{1 \oplus \zeta^{-1}\gamma|_{A/X}\zeta \mid \gamma \in \Gamma\}$. Then:*

- 1) *if $\Gamma_1 \leq \Gamma$, then $\Gamma = C_\Gamma(X) \rtimes \Gamma_1$ and $C_\Gamma(A/X) = \Sigma \rtimes \Gamma_1$;*
- 2) *if $\Gamma_2 \leq \Gamma$, then $\Gamma = C_\Gamma(A/X) \rtimes \Gamma_2$ and $C_\Gamma(X) = \Sigma \rtimes \Gamma_2$;*
- 3) *if $\sigma \in \Sigma$, $\gamma_1 \in C_\Gamma(A/X)$ and $\gamma_2 \in C_\Gamma(X)$, then*

$$\sigma^{\gamma_1 \gamma_2} = \gamma_2^{-1}(\sigma - 1)\gamma_1 + 1.$$

In particular, if $\Gamma_1 \Gamma_2 \leq \Gamma$, then $\Gamma = \Sigma \rtimes (\Gamma_1 \times \Gamma_2)$. \square

Proposition 4.4. *Let A be an abelian group and $T := T(A)$.*

- 1) *If $r_0(A) < \infty$, then the automorphisms induced by $\text{FAut}(A)$ via conjugation on $\text{St}(A, T)$ are finitary;*
- 2) *If $r_0(A) = \infty$ and the quotient A/T is free abelian, then there is $\gamma \in \text{FAut}(A)$ which induces via conjugation on $\text{St}(A, T)$ a non-finitary automorphism, provided $\text{FAut}(T) \neq 1$.*

Proof. 1) Denote $\bar{A} = A/T$ and fix $\gamma \in \text{FAut}(A)$. By Fact 4.1, for each $\sigma \in \text{St}(A, T)$ we have $[\sigma, \gamma]\mathcal{H} = (\sigma^{-1}\sigma^\gamma)\mathcal{H} = -(\sigma - 1) + (\sigma^\gamma - 1) = -(\sigma - 1) + (\sigma - 1)\gamma = (\sigma - 1)(\gamma - 1) =: \varphi_\sigma$. Thus we have to check that the set $\{\varphi_\sigma \mid \sigma \in \text{St}(A, T)\}$ is finite. For each σ , we have that $\text{im}(\varphi_\sigma) \leq \text{im}(\gamma - 1)$ has finite order, say n . On the other hand, $\ker(\varphi_\sigma) \geq n\bar{A}$ and $\bar{A}/n\bar{A}$ is finite since \bar{A} has finite rank.

2) If $A = T \oplus K$, where K is free abelian on the infinite \mathbb{Z} -basis $\{a_i\}$, take $\gamma_0 \in \text{FAut}(T) \setminus \{1\}$. Let $t \in T$ such that $t\gamma_0 \neq t$ and $\gamma := \gamma_0 \oplus 1$. For each i define $\sigma_i \in \text{St}(A, T)$

by the rule $a_i(\sigma_i - 1) := t$ and $a_j(\sigma_i - 1) := 0$ if $j \neq i$. Then there are infinitely many $[\sigma_i, \gamma]$, as $a_i \notin \ker([\sigma_i, \gamma]\mathcal{H}) \ni a_j$ for each $i \neq j$. \square

Clearly, it may well happen that $\text{St}(A, T) \not\leq \text{FAut}(A)$, as in the case $A = \mathbb{Z}(p^\infty) \oplus \mathbb{Q}_p$. On the other hand, we do have $\text{St}(A, X) \leq \text{FAut}(A)$, provided that one of the following holds:

- A/X is bounded and X has finite rank, as in Propositions 4.5 and 5.1.(2);
- A/X has finite rank and X is bounded, as in Proposition 6.3;
- A/X is finitely generated and X is periodic, as in Proposition 6.4.

Here we consider an instance of the first case with $X = D(A)$ and prove a proposition concerning finitary automorphisms.

Proposition 4.5. *Let A be an abelian p -group such that $D := D(A)$ has finite rank and A/D is bounded. Then $\Sigma := \text{St}(A, D)$ is a bounded abelian p -group and there is a subgroup $\Phi \simeq \text{FAut}(A/D)$ such that*

$$\text{FAut}(A) = \Sigma \rtimes \Phi$$

where the automorphisms induced by Φ via conjugation on Σ are finitary and this action is faithful.

Proof. First note that if $\sigma \in \Sigma$, then $[A, \sigma] = A(\sigma - 1)$ is finite, since it is both finite rank and bounded. Hence $\sigma \in \text{FAut}(A)$. Consider a decomposition $A = D \oplus B$ and apply Lemma 4.3, with $X = D$ and $\Gamma = \text{FAut}(A) = C_\Gamma(X)$. Put $\Phi := \Gamma_2$. Then $\text{FAut}(A) = \Sigma \rtimes \Phi$, as claimed.

Let $\gamma \in \Phi$. We have to show that set $\{[\sigma, \gamma] \mid \sigma \in \Sigma\}$ is finite. As in Proposition 4.4, we have $[\sigma, \gamma]\mathcal{H} = (\sigma^{-1}\sigma^\gamma)\mathcal{H} = (1 - \sigma) + (\sigma^\gamma - 1) = -(\sigma - 1) + \gamma^{-1}(\sigma - 1) = (\gamma^{-1} - 1)(\sigma - 1) =: \varphi_\sigma$. Thus we have to count how many homomorphisms φ_σ there are. On the one hand, $\ker(\varphi_\sigma)$ contains $\ker(\gamma^{-1} - 1)$ which has finite index in A/D . On the other hand, the image of each φ_σ is contained in the finite subgroup $D[p^m]$, where p^m is a bound for A/D . Therefore, there are only finitely many φ_σ , once γ is fixed.

Let us check that the action is faithful. Let $1 \neq \gamma \in \Phi$ and let $b \in B$ with maximal order and $b \neq b\gamma$. Then $B = \langle b \rangle \oplus B_0$ and we can write $b\gamma = nb + b_0$ with $n \in \mathbb{Z}, b_0 \in B_0$. If $b \neq nb$, then there is $\sigma \in \Sigma$ such that $B_0(\sigma - 1) = 0$ and $b(\sigma - 1) = d$ where $d \in D$ has the same order as b . Thus, by Fact 4.1, $b\gamma(\sigma^\gamma - 1) = b\gamma(\gamma^{-1}(\sigma - 1)) = d$, while $b\gamma(\sigma - 1) = nd$. Therefore $\sigma^\gamma \neq \sigma$. Similarly, if $b = nb$, then there is $\sigma \in \Sigma$ such that $b(\sigma - 1) = 0$ and $b_0(\sigma - 1) = d_1$ of order p . Then $b\gamma(\sigma^\gamma - 1) = 0$, while $b\gamma(\sigma - 1) = d_1$ and again $\sigma^\gamma \neq \sigma$. \square

Remark 4.6. In Proposition 4.5, Σ need not be contained in the FC-center of $\text{FAut}(A)$.

Proof. Write $A = D \oplus B_0$ where $D \simeq \mathbb{Z}(p^\infty)$ and $B_0 = \bigoplus_i \langle b_i \rangle \leq B$ is infinite and homogeneous. Fix $\sigma \in \Sigma$ such that $b_1(\sigma - 1) = d$, where d is an element of D of order p , and $\sigma - 1 = 0$ on $D \oplus \bigoplus_{j \neq 1} \langle b_j \rangle$. For each i consider $\gamma_i \in \text{FAut}(A)$ switching $b_i \leftrightarrow b_1$ and acting trivially on $D \oplus (\bigoplus_{j \notin \{1, i\}} \langle b_j \rangle)$. Then $\sigma_i^\gamma = \gamma_i^{-1}(\sigma - 1) + 1$. Hence $b_i \sigma_i^{\gamma_i} = d + b_i$ and $b_j \sigma_i^{\gamma_i} = b_j$ for each $j \neq i$. \square

Now an instance of a similar argument with $X = T(A)$

Proposition 4.7. Let A be an abelian group with $r_0(A) < \infty$ such that A/T is finitely generated (resp. $T := T(A)$ is bounded). Then $\Sigma := \text{St}(A, T)$ is a periodic (resp. bounded) abelian group and there is a subgroup $\Phi_1 \simeq \text{FAut}(T)$ such that

$$\text{FAut}(A) = \Sigma \rtimes \Phi_1$$

where Φ_1 induces via conjugation on Σ finitary automorphisms.

If $A/T \neq 0$ is finitely generated, then this action is faithful, while if $A = \mathbb{Z}_{12} \oplus \mathbb{Q}_{(2)}$ it is not.

Proof. In any case, we can write $A = T \oplus K$ where $r := r_0(K) < \infty$. Recall that $\Sigma \simeq \text{Hom}(A/T, T)$. Note that $\Sigma \leq \text{FAut}(A)$. In fact, if $\sigma \in \Sigma$, then $\sigma - 1 \in \text{Hom}(A/T, T)$ and $A(\sigma - 1)$ is an abelian group which is both finitely generated and periodic (resp. finite rank and bounded). Hence $A(\sigma - 1)$ is finite that is $\sigma \in \text{FAut}(A)$.

Clearly $\Phi_1 := \{\varphi \oplus 1 \mid \varphi \in \text{FAut}(T)\} \simeq \text{FAut}(T)$ and $\Phi_1 \leq \text{FAut}(A)$. By Lemma 4.3.(1) we have that $\text{FAut}(A) = \Sigma \rtimes \Phi_1$. By Proposition 4.4, Φ_1 induces via conjugation on Σ finitary automorphisms.

If A/T is finitely generated, then $\Sigma \simeq \text{Hom}(A/T, T)$ is a periodic abelian group which is naturally isomorphic to the direct sum of r copies of T as a right $\text{Aut}(A)$ -module. Therefore the action of Φ_1 on Σ is faithful. Finally, if $A = \mathbb{Z}_{12} \oplus \mathbb{Q}_{(2)}$, we have $\Phi_1 \simeq \mathcal{U}\mathbb{Z}_{12}$ and $\Sigma \simeq \mathbb{Z}_3$; hence the action is not faithful. \square

5 The group $\text{IAut}(A)$, when A is periodic

To give a detailed description of $\text{IAut}(A)$ when A a p -group, let us introduce some terminology. By *essential exponent* $e = \text{eexp}(A)$ of A we mean the smallest e such that $p^e A$ is finite, or $e = \infty$ if A is unbounded. In the former case, this is equivalent to saying

that $A = A_0 \oplus A_1 \oplus A_2$ where A_0 is finite, $\exp(A_1) < e \leq \exp(A_0)$ and A_2 is the sum of infinitely many cyclic groups of order p^e . In [7] we called *critical* a p -group of type $A = B \oplus D$ with B infinite but bounded and $D \neq 0$ divisible with finite rank (see Lemma 2.1.(4)). Critical groups will be a tool to describe $\text{IAut}(A)$ when A is periodic.

Proposition 5.1. *Let A be an abelian p -group and $D := D(A)$.*

1) *If A is non-critical, then $\text{IAut}(A) = \text{PAut}(A) \cdot \text{FAut}(A)$ where $\text{PAut}(A) \cap \text{FAut}(A)$ is either trivial or cyclic of order p^{m-e} , according as A is unbounded or $m := \exp(A) < \infty$ and $e := e\exp(A)$.*

2) *If $A = D \oplus B$ is critical, let $\Delta := \{1 \oplus n \mid n \in \mathbb{Z} \setminus p\mathbb{Z}\}$, $\Phi := \{1 \oplus \varphi_0 \mid \varphi_0 \in \text{FAut}(B)\}$ and $\Psi := \{1 \oplus \gamma_0 \mid \gamma_0 \in \text{IAut}(B)\}$, then*

$$\text{IAut}(A) = \text{PAut}(A) \times (\text{FAut}A \cdot \Delta).$$

Moreover $\text{FAut}A \cdot \Delta = C_{\text{IAut}(A)}(D) = \Sigma \rtimes \Psi$, where $\text{FAut}(A) = \Sigma \rtimes \Phi$ and

i) $\Sigma := \text{St}(A, D)$ is an infinite abelian p -group, $\exp(\Sigma) = \exp(B) =: m' < \infty$ and $e\exp(\Sigma) = e\exp(B) =: e'$;

ii) $\Psi = \Phi\Delta \simeq \text{IAut}(B)$ where $[\Phi, \Delta] = 1$ and Ψ induces via conjugation on Σ inertial automorphisms and this action is faithful;

iii) $\Delta \simeq \text{PAut}(B) \simeq \mathcal{U}(\mathbb{Z}(p^{m'}))$, each $\delta_n := 1 \oplus n \in \Delta$ acts via conjugation on Σ as the multiplication by n and $\text{FAut}(A) \cap \Delta$ has order $p^{m'-e'}$;

iv) $\Phi \simeq \text{FAut}(B)$ and Φ induces via conjugation on Σ finitary automorphisms.

Proof. Let $\gamma \in \Gamma := \text{IAut}(A)$.

1) If A is non-critical, then, according to Lemma 2.1.(4), there exist a p -adic α and a subgroup A_0 of finite index in A such that $\gamma|_{A_0} = \alpha$. Thus $\gamma^{-1}\alpha$ acts on A_0 as the identity map, that is, $\gamma^{-1}\alpha \in \text{FAut}(A)$. Hence $\text{IAut}(A) = \text{PAut}(A) \cdot \text{FAut}(A)$. Further, if the p -adic number β is in $\text{PAut}(A) \cap \text{FAut}(A)$, then β is trivial on a subgroup B of finite index in A . Therefore $\beta = 1$, provided $\exp(A) = \infty$. Otherwise, $\exp(B) \geq e$ and $\beta \equiv 1 \pmod{p^e}$. Thus there are at most p^{m-e} choices for such a β . On the other hand, each p -adic number $\beta \equiv 1 \pmod{p^e}$ is finitary.

2) Let $A = D \oplus B$ be critical. By Lemma 2.1.(4) there exists an invertible p -adic α such that $\gamma|_D = \alpha$. Thus $\gamma_1 := \gamma\alpha^{-1} \in C_\Gamma(D)$. Clearly, $\text{PAut}(A) \cap C_\Gamma(D) = 1$, so that $\text{IAut}(A) = \text{PAut}(A) \times C_\Gamma(D)$.

Again by Lemma 2.1.(4), γ_1 acts by multiplication by an integer n on a subgroup of finite index in $A[p^{m'}]$ where $A[p^{m'}] \geq B$. Therefore, if $\delta_n := 1 \oplus n \in \Delta$ with respect to $A = D \oplus B$, we have $\gamma_1\delta_n^{-1} \in \text{FAut}(A)$. Hence $C_\Gamma(D) = \text{FAut}(A) \cdot \Delta$.

It is routine to verify that (i) holds, since $\Sigma := \text{St}(A, D) \simeq \text{Hom}(B, D)$. By Proposition 4.5, (iv) holds as well. By Lemma 4.3 (with $X := D$, $K := B$ and so $\Gamma_2 = \Psi$), we have $C_\Gamma(D) = \Sigma \rtimes \Psi$ as stated in (2). Then, applying part (1) of the statement to B , we have $\Psi = \Delta\Phi$ and $[\Phi, \Delta] = 1$ as in (ii). Moreover, $\text{FAut}(A) \cap \Delta$ has order $p^{m'-e'}$.

By Lemma 4.2, we have that Δ acts on Σ as in (iii). Thus the whole $\Psi = \Phi\Delta$ acts via conjugation on Σ inducing inertial automorphisms and (ii) holds.

It remains to show that Ψ acts faithfully on Σ . Let $\varphi\delta_n \in C_\Psi(\Sigma)$ with $\varphi \in \Phi$ and $\delta_n := 1 \oplus n \in \Delta$. On the one hand, δ_n acts via conjugation on Σ as the multiplication by n by (iii). On the other hand, δ_n is finitary on Σ by (iv). Since $\text{eexp}(\Sigma) = \text{eexp}(B)$ by (i), then multiplication by n is finitary on B . Thus $\delta_n \in C_\Phi(\Sigma) = 1$ by Proposition 4.5. \square

We have seen that, if A is a p -group, then $\text{IAut}(A)$ is central-by-(locally finite). If A is a critical p -group, one can ask whether there is an abelian normal subgroup Λ of $\text{IAut}(A)$ such that $\text{IAut}(A) = \Lambda \cdot \text{FAut}(A)$. The answer is in the negative, as in the following remark. First we state an easy lemma.

Lemma 5.2. *If B_0 is a subgroup of finite index in a bounded abelian group B , then there are subgroups B_1 and B_2 such that B_2 is finite, $B_1 \geq B_0$ and $B = B_1 \oplus B_2$.*

Proof. Clearly there is a finite subgroup F such that $B = B_0 + F$. Since B_0 is separable and $B_0 \cap F$ is finite, then there is a finite subgroup $B_3 \geq B_0 \cap F$ such that $B_0 = B_1 \oplus B_3$ for some $B_1 \leq B_0$. Fix B_1 and $B_2 := B_3 + F$. On the one hand $B_1 + B_2 = B_1 + B_3 + F = B_0 + F = B$. On the other hand, by Dedekind law, $B_1 \cap B_2 = B_1 \cap (B_3 + F) = B_1 \cap (B_0 \cap (B_3 + F)) = B_1 \cap (B_3 + (B_0 \cap F)) = B_1 \cap B_3 = 0$. \square

Remark 5.3. *If A is a critical p -group (with $p \neq 2$) and $\Lambda \triangleleft \text{IAut}(A)$ is such that $C_\Gamma(D) = \text{FAut}(A) \cdot \Lambda$, then Λ is neither finite nor locally nilpotent.*

Proof. We use the same notation as in Proposition 5.1. Let $n \in \mathbb{N}$ be a primitive root of 1 mod $p^{m'}$ and consider $\delta := 1 \oplus n \in \Delta$ with respect to $A = D \oplus B$. Since $\Delta \leq C_\Gamma(D) = \text{FAut}(A) \cdot \Lambda$, then we can write with $\varphi \in \text{FAut}(A)$ and $\lambda \in \Lambda$. Hence $\delta = \lambda = n$ on some subgroup B_0 of finite index in B . By Lemma 5.2, $B = B_1 \oplus B_2$ with $B_1 \leq B_0$ and B_2 finite. Put $A_1 := D + B_1$ and note that $\lambda|_{A_1} = 1 \oplus n$ with respect to $A_1 = D \oplus B_1$.

It is sufficient to show that $\langle \lambda \rangle^{\Gamma_1}$ is infinite and not locally nilpotent, where Γ_1 is the group of (inertial) automorphisms of A of type $\gamma_1 \oplus 1$ with respect to $A = A_1 \oplus B_2$, with

$\gamma_1 \in \text{IAut}(A_1)$. Thus we may assume $A_1 = A$ and $\Gamma := \Gamma_1$. Then multiplication by n is in Λ and $\Lambda = \Delta^\Gamma$.

We claim that $\Delta^\Gamma = \Sigma \rtimes \Delta$. In fact, by Proposition 5.1 we have that $\Delta \simeq \mathcal{U}(\mathbb{Z}_{p^{m'}})$ acts faithfully by multiplication on the infinite abelian p -group Σ of exponent m' and then $\Sigma = [\Sigma, \Delta]$ and $\Delta^\Gamma = \Sigma\Delta$, as claimed. Thus Δ^Γ is not locally nilpotent, since the action of Δ on Σ is fixed-point-free. \square

Proof of Theorem A. By Lemma 2.1.(3), $\text{IAut}(A)$ may be identified with $\text{PAut}(A) \cdot \text{Dr}_p \text{IAut}(A_p)$. Apply Proposition 5.1 to each A_p . Let π be the set of primes p for which A_p is not critical. If $p \in \pi$, we have $\text{IAut}(A_p) = \text{PAut}(A_p) \cdot \text{FAut}(A_p)$. Otherwise, for each $p \notin \pi$, there are subgroups $\Delta_p, \Sigma_p, \Psi_p$ corresponding to Δ, Σ, Ψ in Proposition 5.1 such that $\text{IAut}(A_p) = \text{PAut}(A_p) \cdot \text{FAut}(A_p) \cdot \Delta_p$ and $\text{FAut}(A_p) \cdot \Delta_p = \Sigma_p \rtimes \Psi_p$. Now it is routine to verify that the statement follows by setting $\Delta := \text{Dr}_{p \notin \pi} \Delta_p$, $\Sigma := \text{Dr}_{p \notin \pi} \Sigma_p$, $\Psi := \text{Dr}_{p \notin \pi} \Psi_p$, and recalling that $\text{Dr}_p \text{FAut}(A_p) = \text{FAut}(\text{Dr}_p A_p)$. \square

Remark that, in Theorem A, when we consider the action of the above Ψ on the p -component Σ_p of Σ we are concerned with subgroups of $\text{IAut}(\Sigma_p) = \text{PAut}(\Sigma_p) \cdot \text{FAut}(\Sigma_p)$, where Σ_p is a bounded abelian p -group and $\text{PAut}(\Sigma_p)$ is finite abelian.

6 The group $\text{IAut}(A)$, when A is non-periodic

To prove Theorem B we point out the existence of some inertial automorphisms of a particular type.

Lemma 6.1. *Let A be a non-periodic abelian group and $\pi_*(A)$ be the set of primes such that A/A_p is p -divisible and one of the following holds:*

- A_p is finite,
- $r_0(A)$ is finite and A_p is bounded.

Then, for each $p \in \pi_(A)$, there is a unique $C^{(p)}$ such that $A = A_p \oplus C^{(p)}$ and the automorphism $\gamma_{(p)} := 1 \oplus p$ (with respect to this decomposition) is inertial.*

Moreover, the subgroup $Q(A) := \langle \gamma_{(p)} \mid p \in \pi_(A) \rangle \times \{\pm 1\}$ is a central subgroup of $\text{IAut}(A)$ which is isomorphic to a multiplicative group of rational numbers. Furthermore, $\text{IAut}_1(A) \cap Q(A) = 1$.*

Proof. The proof of the first part of the statement, concerning $C^{(p)}$, is routine. Let us show that $\gamma_{(p)}$ is inertial. For each $H \leq A$ we have $H + H\gamma_{(p)} \leq H + A_p$. If A_p is finite, then $(H + H\gamma_{(p)})/H$ is finite as well and $\gamma_{(p)}$ is inertial. If $r_0(A)$ is finite and A_p

is bounded, let V be the $\langle \gamma_{(p)} \rangle$ -closure of a free subgroup of C with maximal rank. Then V is torsion-free, C/V is a p' -group and $\gamma_{(p)}$ acts as a power automorphism on A/V . Thus we apply Lemma 2.1.(2) and deduce that $\gamma_{(p)}$ is inertial. The remaining part of the statement follows straightforward. \square

Proof of Theorem B. Let $\gamma_{(p)}$ and $Q := Q(A)$ as in Lemma 6.1.

We first consider the case when $r_0(A) = \infty$. Let $\gamma \in \text{IAut}(A)$. Then, by Corollary B in [10], we have $\gamma = \gamma_1 \gamma_2^{-1}$ with γ_1, γ_2 inertial. Further, by Lemma 2.1.(1), there is a subgroup A_0 with finite index in A such that we have $\gamma|_{A_0} = m/n = p_1^{s_1} \cdots p_t^{s_t} \in \mathbb{Q}$ (m, n coprime, p_i prime, $s_i \in \mathbb{Z}$). Also $\text{IAut}_1(A) = \text{FAut}(A)$ and $\gamma = m/n$ on A/T as well. If $m/n = 1$, then $\gamma \in \text{FAut}(A)$. If $m/n = -1$, put $\gamma_0 := -1 \in Q$. Otherwise, since γ is invertible, $mA_0 = A_0 = nA_0$. Then for each $p_i \in \pi := \pi(mn)$, the p_i -component of A is finite and A/T is p_i -divisible. Consider then $\gamma_0 := \gamma_{(p_1)}^{s_1} \cdots \gamma_{(p_t)}^{s_t} \in Q$. In both cases, $\gamma \gamma_0^{-1} = 1$ on $A_0/(A_0)_\pi$ hence $\gamma \gamma_0^{-1} \in \text{FAut}(A)$. Thus $\text{IAut}(A) = \text{IAut}_1(A) \times Q(A)$. Moreover, (i) and (ii) are true with $\Gamma = 1$, since $\text{IAut}_1(A) = \text{FAut}(A)$ is locally finite.

Let then $r_0(A) < \infty$ and $\gamma \in \text{IAut}(A)$. By Corollary B in [10] γ is inertial. By Lemma 2.1.(2), we have that $\gamma = m/n = p_1^{s_1} \cdots p_t^{s_t} \in \mathbb{Q}$ (m, n coprime, p_i prime, $s_i \in \mathbb{Z}$) on A/T . We also have that, for each $p_i \in \pi := \pi(mn)$, the group A/T is p_i -divisible and A_{p_i} is bounded. Consider $\gamma_0 := \gamma_{(p_1)}^{s_1} \cdots \gamma_{(p_t)}^{s_t} \in Q$. Clearly $\gamma_0 = m/n$ on A/T . Thus $\gamma \gamma_0^{-1}$ acts trivially on A/T and $\text{IAut}(A) = \text{IAut}_1(A) \times Q(A)$, as stated.

Let Γ be the preimage of $\text{PAut}(T)$ under the canonical homomorphism $\text{IAut}_1(A) \mapsto \text{IAut}(T)$. Now (i) holds, since $\text{IAut}_1(A)/\Gamma$ is locally finite by Theorem A. To check (ii) consider that the derived subgroup Γ' of Γ stabilizes the series $0 \leq T \leq A$ and therefore is abelian. Moreover, by Theorem B in [10], the subgroup Γ' consists of finitary automorphisms. Thus Γ' is torsion and (ii) holds by Lemma 4.2. \square

Let us see that there are groups A with few inertial automorphisms even if $r_0(A) < \infty$ and that the canonical homomorphism $\text{IAut}_1(A) \rightarrow \text{IAut}(T)$ need not be surjective.

Proposition 6.2. *Let A be a π -divisible non-periodic abelian group, where π is a set of primes. If $T := T(A)$ is a π -group, then $\text{IAut}_1(A) = 1$.*

Proof. If $r_0(A) = \infty$ then $\text{IAut}_1(A) = \text{FAut}(A)$. Moreover, if $\gamma \in \text{FAut}(A)$ then $A(\gamma - 1)$ is a finite π -group. Then $A/\ker(\gamma - 1)$ is such. Hence $A = \ker(\gamma - 1)$ and $\text{FAut}(A) = 1$.

If $r_0(A) < \infty$, by Lemma 2.1.(2) we have $\gamma = 1$ on some free abelian subgroup $V \leq A$ such that A/V is periodic. Moreover, the π -component B/V of A/V is divisible. Then, by Lemma 2.1, part (3) and (4), we have that γ is a multiplication on B/V . Furthermore,

the group $B/(V+T)$ is π -divisible and has non-trivial p -component for each $p \in \pi$, since $(V+T)/T \simeq V$ is free abelian. Thus from $\gamma = 1$ on B/T it follows that $\gamma = 1$ on $\gamma = 1$. Hence γ stabilizes the series $0 \leq V \leq B$. However $\text{Hom}(B/V, V) = 0$. Then $\gamma = 1$ on B . Therefore $\gamma - 1$ induces a homomorphism $A/B \rightarrow T$ which is necessarily 0 since A/B is a π' -group. Thus $\gamma = 1$ on the whole group A . \square

Proof of Theorem C. It follows from the next propositions which considers cases in which $\text{IAut}_1(A)$ splits on $\Sigma := \text{St}(A, T)$. \square

Proposition 6.3. *Let A be an abelian group and $T := T(A)$.*

If $r_0(A) < \infty$ and T is bounded, then $\Sigma := \text{St}(A, T)$ is a bounded abelian group and there is a subgroup Γ_1 of $\text{IAut}_1(A)$ such that $\Gamma_1 \simeq \text{IAut}(T)$ and

$$\text{IAut}_1(A) = \Sigma \rtimes \Gamma_1$$

where Γ_1 induces via conjugation on Σ inertial automorphisms.

Proof. We can write $A = T \oplus K$ where $r := r_0(K) < \infty$. Note that the group $\Sigma \simeq \text{Hom}(A/T, T)$ is a periodic abelian group which is bounded as T .

Clearly $\Gamma_1 := \{\gamma \oplus 1 \mid \gamma \in \text{IAut}(T)\} \simeq \text{IAut}(T)$. If $\gamma \in \text{IAut}(T)$, then $\gamma \oplus 1$ (with respect to $T \oplus K$) is inertial by Lemma 2.1.(4), and so $\Gamma_1 \leq \text{IAut}_1(A)$. Thus we may apply Lemma 4.3 with $\Gamma := \text{IAut}_1(A)$. We obtain $\text{IAut}_1(A) = \Sigma \rtimes \Gamma_1$, as claimed.

By Proposition 5.1, we have $\text{IAut}(T) = \text{FAut}(T) \cdot \text{PAut}(T)$. Hence $\Gamma_1 = \Phi_1 \Delta_1$ where $\Phi_1 := \{\varphi \oplus 1 \mid \varphi \in \text{FAut}(T)\} \simeq \text{FAut}(T)$ acts conjugation on Σ by means of finitary automorphisms, by Proposition 4.7 and $\Delta_1 := \{\delta \oplus 1 \mid \delta \in \text{PAut}(T)\} \simeq \text{PAut}(T)$ acts via conjugation on Σ by means of multiplications, by Lemma 4.2. Therefore the whole Γ_1 induces via conjugation on Σ inertial automorphisms. \square

We notice that the action of Γ_1 on Σ in Proposition 6.3 need not be faithful, as already seen in Proposition 4.7.

Proposition 6.4. *Let A be a non periodic abelian group and $T := T(A)$.*

If A/T is finitely generated, then $\Sigma := \text{St}(A, T)$ is a periodic abelian group and there is a subgroup Γ_1 of $\text{IAut}_1(A)$ such that $\Gamma_1 \simeq \text{IAut}(T)$ and

$$\text{IAut}_1(A) = \Sigma \rtimes \Gamma_1$$

where Γ_1 induces via conjugation on Σ inertial automorphisms and this action is faithful.

If in addition T is unbounded, then $\text{IAut}_1(A)$ is not nilpotent-by-(locally finite). Further, if $A_{2'}$ is unbounded, then $\text{IAut}_1(A)$ is not even (locally nilpotent)-by-(locally finite).

Proof. As in the proof of Proposition 6.3, we can write $A = T \oplus K$ where K is finitely generated. The group $\Sigma \simeq \text{Hom}(A/T, T)$ is a periodic abelian group which is isomorphic to the direct sum $\oplus_r T$ of $r := r_0(A) > 0$ copies of T as a right $\text{Aut}(A)$ -module.

Clearly $\Gamma_1 := \{\gamma \oplus 1 \mid \gamma \in \text{IAut}(T)\} \simeq \text{IAut}(T)$. If $\gamma \in \text{IAut}(T)$, then $\gamma \oplus 1$ (with respect to $T \oplus K$) is inertial by Lemma 2.1.(4). Hence $\Gamma_1 \leq \text{IAut}_1(A)$. Thus we may apply Lemma 4.3 with $\Gamma := \text{IAut}_1(A)$, and we obtain $\text{IAut}_1(A) = \Sigma \rtimes \Gamma_1$.

Let us investigate now the action of Γ_1 via conjugation on Σ . Assume first that T is a p -group. Let $\gamma \in \text{IAut}(T)$. By Proposition 5.1, $\gamma = \gamma_0 \varphi$, where $\varphi \in \text{FAut}(T)$ and either $\gamma_0 \in \text{PAut}(T)$ or T is a critical p -group and γ_0 induces multiplications on both $D(T)$ and $T/D(T)$. Recall that Σ is $\text{Aut}(A)$ -isomorphic to $\oplus_r T$. In the former case, that is if $\gamma_0 \in \text{PAut}(T)$, then $\gamma_0 \oplus 1$ acts via conjugation on Σ as a power automorphism (that is a multiplication). In the latter case, Σ is critical as well and $\gamma_0 \oplus 1$ induces invertible multiplications on both $D(\Sigma)$ and $\Sigma/D(\Sigma)$. Thus $\gamma_0 \oplus 1$ acts via conjugation on Σ as an inertial automorphism of Σ , by Lemma 2.1.(4). In both cases, by Proposition 4.7, φ acts via conjugation on Σ as a finitary automorphism. Hence $\gamma \oplus 1$ acts via conjugation on Σ as an inertial automorphism.

In the general case, when T is any periodic group and $\gamma \in \text{IAut}(T)$, then $\gamma \oplus 1$ (with respect to $T \oplus K$) acts via conjugation as an inertial automorphism on all primary components Σ_p of Σ , by what we have seen above and the fact that $\Sigma_p \simeq \text{Hom}(A/T, A_p)$. Similarly, since $\gamma \oplus 1$ acts as a multiplication on all but finitely many primary components A_p of A , it acts the same way on all but finitely many Σ_p . Thus $\gamma \oplus 1$ is inertial on Σ by Lemma 2.1.(3).

It is clear that the action via conjugation of Γ_1 on Σ is faithful as the standard action of Γ_1 on T is such.

To prove the last part of the statement, note that in the case when T is unbounded, then there exists a non-periodic multiplication α of T . Note that the automorphism $\mu := \alpha \oplus 1$ (with respect to $T \oplus K$) belongs to Γ_1 . If, by the way of contradiction, $\langle \Sigma, \mu \rangle$ is nilpotent-by-(locally finite), then there is $s \in \mathbb{Z} \setminus \{0\}$ such that $\langle \Sigma, \mu^s \rangle$ is nilpotent, so there is $n \in \mathbb{N}$ such that $[\Sigma, {}_n \mu^s] = 0$, and hence $0 = \Sigma(\mu^s - 1)^n = \Sigma(\alpha^s - 1)^n$. This is a contradiction, since Σ is unbounded as T is.

Finally, if $A_{2'}$ is unbounded, then $\Sigma_{2'}$ is unbounded as well. Let α be a non-periodic multiplication of $A_{2'}$. Then, $\mu := \alpha \oplus 1 \oplus 1$ with respect to $A = A_{2'} \oplus A_2 \oplus K$ acts as non-periodic multiplication (by α) of $\Sigma_{2'}$ acting fixed-point-free on a primary component. Thus μ (and any non-trivial power of μ as well) does not belong to the locally nilpotent radical R of $\text{IAut}_1(A)$. Therefore $\text{IAut}_1(A)/R$ is not locally finite. \square

Finally, we note that, despite the above propositions, in the general case the group $\text{IAut}_1(A)$ may be large.

Remark 6.5. *There exists an abelian group A with $r_0(A) = 1$ and $A_p \simeq \mathbb{Z}(p)$ for each prime p such that $\text{IAut}(A) = \text{IAut}_1(A) \times \{\pm 1\}$, $\text{IAut}_1(A) = \Sigma \cdot \text{FAut}(A)$, where $\Sigma := \text{St}_{\text{IAut}(A)}(A, T(A)) \not\leq \text{FAut}(A)$, $\Sigma \simeq \prod_p \mathbb{Z}(p)$ and $\text{IAut}_1(A)/\text{FAut}(A) \simeq \Sigma/T(\Sigma)$ is a divisible torsion-free abelian group with cardinality 2^{\aleph_0} .*

Moreover any element of $\text{IAut}_1(A)$ induces on T a finitary automorphism.

Proof. As in Proposition A in [10], we consider the group $G := B \oplus C$ where $B := \prod_p \langle b_p \rangle$, $C := \prod_p \langle c_p \rangle$, and b_p, c_p have order p, p^2 resp. and p ranges over all primes. Consider then the (aperiodic) element $v := (b_p + pc_p)_p \in G$ and $V := \langle v \rangle$. We have that for each prime p there is an element $d_{(p)} \in G$ such that $pd_{(p)} = v - b_p$. Let $A := V + \langle d_{(p)} \mid p \rangle$. Then $A/T \simeq \langle 1/p \mid p \rangle \leq \mathbb{Q}$, since A/T has torsion free rank 1 and $v + T$ has p -height 1 for each p . Thus $T = T(B) \simeq \bigoplus_p \mathbb{Z}(p)$ and the p -component of A/V is generated by $d_{(p)} + V$ and has order p^2 , since $pd_{(p)} = v - b_p$.

Then $\Sigma \simeq \prod_p \mathbb{Z}(p)$ and $\Sigma \cap \text{FAut}(A) = T(\Sigma)$, hence $\Sigma \not\leq \text{FAut}(A)$. Moreover $A = \langle d_{(p)} \rangle + V$, where $V = \langle v \rangle$ is infinite cyclic and $A_p = \langle b_p \rangle$ has order p . Also $\text{Aut}(A/T) = \{\pm 1\}$ and $\text{IAut}(A) = \text{IAut}_1(A) \times \{\pm 1\}$.

We claim that *if $\gamma \in \text{IAut}_1(A)$ induces on T a finitary automorphism, then $\gamma \in \Sigma \cdot \text{FAut}(A)$* . In fact, $T\gamma$ is finite, so it is a π -component of A for some finite π . Thus $\gamma\gamma_0^{-1} \in \Sigma$, where $\gamma_0 := \gamma|_{A_\pi} \oplus 1$ with respect to $A = A_\pi \oplus K$ and clearly $\gamma_0 \in \text{FAut}(A)$.

Finally we prove the last part of the statement, from which it follows $\text{IAut}_1(A) = \Sigma \cdot \text{FAut}(A)$. Let $\gamma \in \text{IAut}_1(A)$ and $\varphi := \gamma - 1$. Since $A\varphi \leq T$, there exists an integer $n \neq 0$ such that $(nv)\varphi = 0$. We prove that $T\varphi \subseteq A_{\pi(n)}$, which is finite. For any prime p , on the one hand, $nd_{(p)}$ is a p -element modulo $\langle nv \rangle \leq \ker \varphi$, hence $(nd_{(p)})\varphi \in A_p$, that implies $(pnd_{(p)})\varphi = p(nd_{(p)})\varphi = 0$. On the other hand, $(pnd_{(p)})\varphi = n(v - b_p)\varphi = -n(b_p)\varphi$. Hence, if $p \notin (n)\varphi$, then $A_p\varphi = 0$. \square

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